

Faddeev-Jackiw quantization of four dimensional BF theory

Alberto Escalante* and Prihel Cavildo Sánchez†

*Instituto de Física, Universidad Autónoma de Puebla,
Apartado Postal J-48 72570, Puebla Pue., México.*

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The symplectic analysis of a four dimensional BF theory in the context of the Faddeev-Jackiw symplectic approach is performed. It is shown that this method is more economical than Dirac's formalism. In particular, the complete set of Faddeev-Jackiw constraints and the generalized Faddeev-Jackiw brackets are reported. In addition, we show that the generalized Faddeev-Jackiw brackets and the Dirac ones coincide to each other. Finally, the similarities and advantages between Faddeev-Jackiw method and Dirac's formalism are briefly discussed.

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I. INTRODUCTION

It is well-known that the topological theories have a relevant role in the context of gravity. In fact, topological theories are good laboratories for testing classical and quantum ideas of generally covariant gauge systems. Topological theories are characterized by lacking of physical degrees of freedom, either in three or four dimensions they have a close relation with General Relativity [GR] just as the background independence and the diffeomorphisms covariance, this is, all the dynamical variables characterizing the theory are dynamical ones. In the three dimensional case a relevant example of topological theory is the Chern-Simons theory. In fact, basically Chern-Simons theory describes GR, it has been showed that these theories are equivalent up to a total derivative [1, 2], and also there exist a relation between these theories defined with (or without) an Immirzi-like parameter [3, 4]. Furthermore, we can find a recent work where the Chern-Simons state describes a topological state with unbroken diffeomorphism invariance in Yang-Mills and GR [5]. In the Loop Quantum Gravity context, that state is called the Kodama state and has been studied in interesting works by Smolin, arguing that the Kodama state at least for the de Sitter spacetime, Loop Quantum Gravity does have a good low energy limit [6]. On the other hand, in four dimensions there exist the so-called BF theory. In fact, BF theories were introduced as generalizations of three dimensional Chern-Simons actions or in other cases, can also be considered as a zero coupling limit of Yang-Mills theories [7, 8]. Moreover, we find in the literature several examples where BF theories with additional extra constraints describe gravity, for instance, the well-known formulations of Plebanski and Macdowell-Mansouri [9, 10]. In addition, within the modern quantization scheme using tools

*Electronic address: aescalan@ifuap.buap.mx

†Electronic address: pcavildo@ifuap.buap.mx

developed in Loop Quantum Gravity, BF theories have been studied in the context of spin foams. In fact, in this approach is not considered the traditional Fock space formalism but holonomies along paths as the basic variables to be quantized [11]. With respect the classical context, there are several works studying the canonical structure of a BF theory, see the for instance the references [12–14]. However, in these works has been used the canonical formalism by using a reduced phase space, this means, it has been considered as dynamical variables only those that occur in the Lagrangian density with temporal derivative, however, in several cases this approach is not convenient, for instance in Palatini’s theory the price to pay for developing the standard approach is that we cannot know the full structure of the constraints and their algebra is not closed [15], thus, the better way to carry out the canonical formalism is by following all Dirac’s steps as it has been commented in [16–21]. In consideration with the commented above, either the classical or the quantum study of BF theories and their close relation with GR is at the present a frontier subject of study [22, 23].

In this manner, with the ideas explained previously in this work the Faddeev-Jackiw [FJ] symplectic quantization of a four-dimensional BF theory is performed. In fact, the FJ method provides an alternative approach for studying constrained systems based on a first-order Lagrangian [24, 25]. The FJ method is a symplectic study and the basic feature of this approach is to treat all the constraints at the same footing. In other words, in FJ method one avoids the classification of the constraints into first-class and second-class ones as in Dirac’s framework is done. In addition, some essential elements of a physical theory such as the degrees of freedom, the gauge symmetry and the quantization brackets called the generalized FJ brackets can also be derived; Dirac’s and generalized FJ brackets coincide to each other. However, it is important to remark that in the canonical formalism we must to work by following all Dirac’s steps in order to compare with the FJ symplectic formalism. In fact, it has been showed that by following all Dirac’s steps, the Dirac results and the FJ ones coincide [26]. In this respect, in this paper we also develop a pure canonical analysis and we compare the obtained results with the FJ ones. We will start with a $SO(3,1)$ invariant four-dimensional BF theory, however, we will break down the Lorentz group in order to work with a compact group, the remaining group will be $SO(3)$. It is important to comment that in [27] a pure canonical analysis of a $SO(3,1)$ invariant BF theory has been performed, however, in that paper the Dirac brackets were not reported. The reason is that by working with the $SO(3,1)$ group either the Dirac or FJ constraints of the theory have not a simple structure and this fact difficulties the construction of such brackets. In this respect, we report the complete structure of the constraints of the theory, then the Dirac and the generalized FJ brackets are computed, we will show that the Dirac brackets and the FJ ones coincide to each other. In this manner, our results complete and extend those reported in the literature.

The paper is organized as follows. In Section II the FJ analysis for a four-dimensional BF theory is performed; we report the complete set of FJ constraints. Moreover, in order to obtain a symplectic tensor we fix the gauge, then the generalized FJ brackets are found. In Section III, we develop a pure canonical analysis of the theory under study. We report the complete structure of the first class and second class constraints and we show that the algebra between the constraints is closed in full

agreement with the canonical rules of Dirac's formulation. Then by introducing the Dirac brackets we eliminate the second class constraints. In Section IV we present some remarks and conclusions.

II. FADDEEV-JACKIW FRAMEWORK FOR BF THEORY

In this section we shall perform the FJ analysis, our laboratory will be given by a four-dimensional BF theory described by the following action

$$S[A_\mu^{IJ}, B_{\alpha\beta}^{KL}] = \Xi \int_M F^{IJ} \wedge B_{IJ}, \quad (1)$$

where Ξ is a constant, $B^{IJ} = \frac{1}{2} B_{\alpha\beta}^{IJ} dx^\alpha \wedge dx^\beta$ is a set of six $SO(3,1)$ valued two forms, the two-form curvature F of the Lorentz connection is defined as usual by $F_{\mu\nu IJ} = \partial_\mu A_{\nu IJ} - \partial_\nu A_{\mu IJ} + A_{\mu IK} A_{\nu J}^K - A_{\mu JK} A_{\nu I}^K$. Here, $I, J, K, \dots = 0, 1, 2, 3$ are internal Lorentz indices that can be raised and lowered by the internal metric $\eta_{IJ} = (-1, 1, 1, 1)$, x^μ are the coordinates that label the points of the four-dimensional manifold M , and $\alpha, \beta, \mu, \dots = 0, 1, 2, 3$ are space-time indices.

By performing the $3+1$ decomposition and breaking the Lorentz group down to $SO(3)$ we obtain the following Lagrangian density

$$\begin{aligned} L = \Xi \int \eta^{abc} & \left[B_{ab}^{0i} \dot{A}_{c0i} + \frac{1}{2} B_{ab}^{ij} \dot{A}_{cij} \right. \\ & + \frac{1}{2} A_{0ij} \left(\partial_c B_{ab}^{ij} + B_{ab}^{il} A_c^j{}_l + 2 B_{ab}^{0i} A_{c0}^j \right) \\ & + A_{00i} \left(\partial_c B_{ab}^{0i} + B_{ab}^{0j} A_c^i{}_j + B_{ab}^{ij} A_{c0j} \right) \\ & + B_{0a}^{0i} \left(\partial_b A_{c0i} - \partial_c A_{b0i} + A_{b0j} A_c^j{}_i + A_{c0}^j A_{bij} \right) \\ & \left. + \frac{1}{2} B_{0a}^{ij} \left(\partial_b A_{cij} - \partial_c A_{bij} + A_{bil} A_c^l{}_j + A_{bi0} A_c^0{}_j - A_{bjl} A_c^l{}_i - A_{bj0} A_c^0{}_i \right) \right] d^3x, \quad (2) \end{aligned}$$

here, $a, b, c, \dots = 1, 2, 3$, $\epsilon^{0abc} = \eta^{abc}$ and $i, j, k, l, \dots = 1, 2, 3$ are lowered and raised with the Euclidean metric $\eta_{ij} = (1, 1, 1)$. By introducing the following variables [28]

$$\begin{aligned} A_{aij} &\equiv -\epsilon_{ijk} A_a^k, \\ A_{0ij} &\equiv -\epsilon_{ijk} A_0^k, \\ B_{abij} &\equiv -\epsilon_{ijk} B_{ab}^k, \\ B_{0aij} &\equiv -\epsilon_{ijk} B_{0a}^k, \\ A_{ai} &\equiv \Upsilon_{ai}, \end{aligned} \quad (3)$$

the Lagrangian takes the following form

$$\mathcal{L} = \Xi \eta^{abc} B_{ab}^{0i} \dot{A}_{c0i} + \Xi \eta^{abc} B_{abi} \dot{\Upsilon}_c^i \quad (4)$$

$$\begin{aligned} & - \left[-A_0^i \left(\Xi \eta^{abc} B_{abi} \right) + \Xi \eta^{abc} \epsilon_{ik}^j B_{abj} \Upsilon_c^k - \Xi \eta^{abc} \epsilon_{jki} B_{ab}^{0j} A_{c0}^k \right) \\ & - A_{00i} \left(\Xi \eta^{abc} B_{ab}^{0i} \right) - \Xi \eta^{abc} \epsilon_{jk}^i B_{ab}^{0j} \Upsilon_c^k - \Xi \eta^{abc} \epsilon^{ijk} B_{abk} A_{c0j} \right) \\ & - \Xi \eta^{abc} B_{0a}^{0i} \left(\partial_b A_{c0i} - \partial_c A_{b0i} + \epsilon_i^{jk} A_{b0j} \Upsilon_{ck} - \epsilon_{ijk} A_{c0}^j \Upsilon_b^k \right) \\ & - \Xi \eta^{abc} B_{0ai} \left(\partial_b \Upsilon_c^i - \partial_c \Upsilon_b^i + \epsilon_{jk}^i \Upsilon_b^j \Upsilon_c^k - \epsilon^{ijk} A_{b0j} A_{c0k} \right) \right]. \quad (5) \end{aligned}$$

In this manner, we can identify the symplectic Lagrangian given by

$$\mathcal{L}^{(0)} = \Xi\eta^{abc}B_{ab}{}^{0i}\dot{A}_{c0i} + \Xi\eta^{abc}B_{abi}\dot{\Upsilon}_c{}^i - \mathcal{V}^{(0)}, \quad (6)$$

where $\mathcal{V}^{(0)}$ is the symplectic potential expressed as

$$\begin{aligned} \mathcal{V}^{(0)} = & -A_0{}^i \left(\partial_c (\Xi\eta^{abc}B_{abi}) + \Xi\eta^{abc}\epsilon_{ik}^j B_{abj}\Upsilon_c{}^k - \Xi\eta^{abc}\epsilon_{jki} B_{ab}{}^{0j} A_{c0}{}^k \right) \\ & -A_{00i} \left(\partial_c (\Xi\eta^{abc}B_{ab}{}^{0i}) - \Xi\eta^{abc}\epsilon_{jk}^i B_{ab}{}^{0j}\Upsilon_c{}^k - \Xi\eta^{abc}\epsilon^{ijk} B_{abk} A_{c0j} \right) \\ & -\Xi\eta^{abc}B_{0a}{}^{0i} \left(\partial_b A_{c0i} - \partial_c A_{b0i} + \epsilon_i{}^{jk} A_{b0j}\Upsilon_{ck} - \epsilon_{ijk} A_{c0}{}^j \Upsilon_b{}^k \right) \\ & -\Xi\eta^{abc}B_{0ai} \left(\partial_b \Upsilon_c{}^i - \partial_c \Upsilon_b{}^i + \epsilon^i{}_{jk} \Upsilon_b{}^j \Upsilon_c{}^k - \epsilon^{ijk} A_{b0j} A_{c0k} \right). \end{aligned} \quad (7)$$

From the symplectic Lagrangian (6) we identify the following symplectic variables $\xi^{(0)} = (A_{a0i}, B_{ab}{}^{0i}, \Upsilon_a{}^i, B_{abi}, A_0{}^i, A_{00i}, B_{0a}{}^{0i}, B_{0ai})$ and the 1-forms $a^{(0)} = (\Xi\eta^{abc}B_{ab}{}^{0i}, 0, \Xi\eta^{abc}B_{abi}, 0, 0, 0, 0, 0)$. In this manner, the symplectic matrix defined as $f_{ij}^{(0)}(x, y) = \frac{\delta a_j(y)}{\delta \xi^i(x)} - \frac{\delta a_i(x)}{\delta \xi^j(y)}$, is given by

$$f_{ij}^{(0)} = \begin{pmatrix} 0 & -\Xi\eta^{abc}\delta_j^i & 0 & 0 & 0 & 0 & 0 & 0 \\ \Xi\eta^{abc}\delta_j^i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Xi\eta^{abc}\delta_j^i & 0 & 0 & 0 & 0 \\ 0 & 0 & \Xi\eta^{abc}\delta_j^i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \delta^3(x-y), \quad (8)$$

we observe that $f_{ij}^{(0)}$ is singular and therefore, there will constraints. The modes of $f_{ij}^{(0)}$ are given by the following 4 vectors

$$v^{(0)}{}_1 = (0, 0, 0, 0, V^{A_0^i}, 0, 0, 0), \quad (9)$$

$$v^{(0)}{}_2 = (0, 0, 0, 0, 0, V^{A_{00i}}, 0, 0), \quad (10)$$

$$v^{(0)}{}_3 = (0, 0, 0, 0, 0, 0, V^{B_{0a}^{0i}}, 0), \quad (11)$$

$$v^{(0)}{}_4 = (0, 0, 0, 0, 0, 0, 0, V^{B_{0ai}}), \quad (12)$$

where $V^{A_0^i}, V^{A_{00i}}, V^{B_{0a}^{0i}}$ and $V^{B_{0ai}}$ are arbitrary functions. Hence, by using these modes we find the following constraints

$$\begin{aligned}
\Omega^{(0)}_{i} &= \int d^3x v^{(0)}_{i} \frac{\delta}{\delta \xi^i} \int d^3y \mathcal{V}^{(0)}(\xi) \\
&= \int d^3x V^{A_0^i} \frac{\delta}{\delta A_0^i} \int d^3y \mathcal{V}^{(0)}(\xi) \\
&= \partial_c (\Xi \eta^{abc} B_{abi}) + \Xi \eta^{abc} \epsilon^j_{ik} B_{abj} \Upsilon_c^k - \Xi \eta^{abc} \epsilon_{jki} B_{ab}^{0j} A_{c0}^k, \\
\Omega^{(0)}_{00i} &= \int d^3x v^{(0)}_{i} \frac{\delta}{\delta \xi^i} \int d^3y \mathcal{V}^{(0)}(\xi) \\
&= \int d^3x V^{A_{00i}} \frac{\delta}{\delta A_{00i}} \int d^3y \mathcal{V}^{(0)}(\xi) \\
&= \partial_c (\Xi \eta^{abc} B_{ab}^{0i}) - \Xi \eta^{abc} \epsilon^i_{jk} B_{ab}^{0j} \Upsilon_c^k - \Xi \eta^{abc} \epsilon^{ijk} B_{abk} A_{c0j}, \\
\Omega^{(0)}_{0a_{0i}} &= \int d^3x v^{(0)}_{i} \frac{\delta}{\delta \xi^i} \int d^3y \mathcal{V}^{(0)}(\xi) \\
&= \int d^3x V^{B_{0a}^{0i}} \frac{\delta}{\delta B_{0a}^{0i}} \int d^3y \mathcal{V}^{(0)}(\xi) \\
&= \Xi \eta^{abc} \left(\partial_b A_{c0i} - \partial_c A_{b0i} + \epsilon_i^{jk} A_{b0j} \Upsilon_{ck} - \epsilon_{ijk} A_{c0}^j \Upsilon_b^k \right), \\
\Omega^{(0)}_{0ai} &= \int d^3x v^{(0)}_{i} \frac{\delta}{\delta \xi^i} \int d^3y \mathcal{V}^{(0)}(\xi) \\
&= \int d^3x V^{B_{0ai}} \frac{\delta}{\delta B_{0ai}} \int d^3y \mathcal{V}^{(0)}(\xi) \\
&= \Xi \eta^{abc} \left(\partial_b \Upsilon_c^i - \partial_c \Upsilon_b^i + \epsilon^i_{jk} \Upsilon_b^j \Upsilon_c^k - \epsilon^{ijk} A_{b0j} A_{c0k} \right), \tag{13}
\end{aligned}$$

we can observe that these constraints are the secondary constraints obtained by using the Dirac method (see the following section). Now we shall observe if there are more constraints, for this aim, we calculate the following system [29]

$$\bar{f}_{kj} \dot{\xi}^{(0)j} = Z_k(\xi), \tag{14}$$

where

$$\bar{f}_{kj} = \begin{pmatrix} f_{ij}^{(0)} \\ \frac{\delta \Omega_i^{(0)}}{\delta \xi^{(0)j}} \end{pmatrix} \quad \text{and} \quad Z_k = \begin{pmatrix} \frac{\delta \mathcal{V}^{(0)}}{\delta \xi^{(0)j}} \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{15}$$

Thus, the symplectic matrix \bar{f}_{ij} is given by

$$\bar{f}_{ij} = \left(\begin{array}{ccc|cccccccc} 0 & -\Xi\eta^{abc}\delta_j^i & 0 & & & & & & \\ \Xi\eta^{abc}\delta_j^i & 0 & 0 & & & & & & \\ 0 & 0 & 0 & & & & & & \\ 0 & 0 & \Xi\eta^{abc}\delta_j^i & & & & & & \\ 0 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & & & & & & \\ 0 & 0 & 0 & & & & & & \\ -\Xi\eta^{abc}\epsilon_{jki}B_{ab}{}^{oj} & -\Xi\eta^{abc}\epsilon_{kji}A_{c0}{}^j & \Xi\eta^{abc}\epsilon_{ik}^jB_{abj} & & & & & & \\ -\Xi\eta^{abc}\epsilon^{ikj}B_{abj} & \Xi\eta^{abc}\left(\delta_k^i\partial_c - \epsilon^i{}_{jk}\Upsilon_c^j\right) & -\Xi\eta^{abc}\epsilon^i{}_{jk}B_{ab}{}^{0j} & & & & & & \\ 2\Xi\eta^{abc}\left(\delta_k^i\partial_c - \epsilon_i{}^{kj}\Upsilon_{bj}\right) & 0 & 2\Xi\eta^{abc}\epsilon_i{}^{jk}A_{b0j} & & & & & & \\ -2\Xi\eta^{abc}\epsilon^{ijk}A_{b0j} & 0 & 2\Xi\eta^{abc}\left(\delta_k^i\partial_b + \epsilon^i{}_{jk}\Upsilon_b^j\right) & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & \\ -\Xi\eta^{abc}\delta_j^i & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & \\ \Xi\eta^{abc}\left(\delta_k^i\partial_c + \epsilon_{ij}^k\Upsilon_c^j\right) & 0 & 0 & 0 & 0 & 0 & & & \\ -\Xi\eta^{abc}\epsilon^{ijk}A_{c0j} & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & \end{array} \right) \delta^3(x-y), \quad (16)$$

The matrix f_{ij} is not a square matrix as expected, however it has null vectors. The null vectors are given by

$$\vec{V}_1 = \left(\epsilon_{kji}A_{c0}{}^jV^i, -\epsilon_{jki}B_{ab}{}^{0j}V^i, \partial_c V^k + \epsilon_{ij}^k\Upsilon_c^jV^i, \epsilon_{ik}^jB_{abj}V^i, 0, 0, 0, 0, V^i, 0, 0, 0 \right), \quad (17)$$

$$\vec{V}_2 = \left(\partial_c V_k - \epsilon^i{}_{kj}\Upsilon_c^jV_i, \epsilon^{ikj}B_{abj}V_i, \epsilon^{kij}A_{c0j}V_i, \epsilon_{ji}^kA_{ab}{}^{0j}V^i, 0, 0, 0, 0, 0, V_i, 0, 0 \right), \quad (18)$$

$$\vec{V}_3 = \left(0, 2\left(\partial_b V^k - \epsilon_i{}^{kj}\Upsilon_{bj}V^i\right), 0, 2\epsilon_i{}^{jk}A_{b0j}V^i, 0, 0, 0, 0, 0, V^i, 0 \right), \quad (19)$$

$$\vec{V}_4 = \left(0, 2\epsilon^{ijk}A_{b0j}V_i, 0, 2\left(\partial_b V^k + \epsilon_{ji}^k\Upsilon_b^jV^i\right), 0, 0, 0, 0, 0, 0, 0, V_i \right). \quad (20)$$

On the other hand, $Z_k(\xi)$ is given by

$$Z_k(\xi) = \begin{pmatrix} \frac{\delta}{\delta \xi^i} \frac{\mathcal{V}^{(0)}(\xi)}{\delta \xi^i} \\ 0 \end{pmatrix} = \begin{pmatrix} \Xi \eta^{abc} \epsilon_j^k A_0^i B_{ab}^{0j} + \eta^{abc} \epsilon^{ikj} A_{00i} B_{abj} + 2\Xi \eta^{abc} \partial_b B_{0a}^{0k} \\ -2\Xi \eta^{abc} \epsilon_i^{kj} B_{0a}^{0i} \Upsilon_{cj} + 2\Xi \eta^{abc} \epsilon^{ijk} B_{0ai} A_{b0j} \\ \Xi \eta^{abc} \epsilon_{ikj} A_0^j A_{c0}^k + \Xi \eta^{abc} \partial_c A_{00i} + \Xi \eta^{abc} \epsilon^j_{ik} A_{00j} \Upsilon_c^k \\ -\Xi \eta^{abc} \epsilon^j_{ik} A_0^i B_{abj} + \Xi \eta^{abc} \epsilon^i_{jk} A_{00i} B_{ab}^{0j} - 2\Xi \eta^{abc} \epsilon_i^{jk} A_{b0j} B_{0a}^{0i} \\ -2\Xi \eta^{abc} \partial_b B_{0ak} - 2\epsilon^i_{jk} \Upsilon_b^j B_{0ai} \\ \Xi \eta^{abc} \partial_c A_0^i - \Xi \eta^{abc} \epsilon^i_{jk} A_0^j \Upsilon_c^k + \Xi \eta^{abc} \epsilon^{jki} A_{00j} A_{c0k} \\ \Omega^{(0)}_i \\ \Omega^{(0)}_{00i} \\ \Omega^{(0)}_{0a0i} \\ \Omega^{(0)}_{0ai} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The contraction of the null vectors with Z_k , namely, $\vec{V}_i^\mu Z_\mu(\xi) = 0$, give identities. For instance, from the contraction of \vec{V}_1 with $Z_k(\xi)$ we obtain

$$\begin{aligned} \vec{V}_1^\mu Z_\mu(\xi) &= \epsilon^j_{ik} A_0^i V^k \left[\partial_c (\Xi \eta^{abc} B_{abj}) + \Xi \eta \epsilon^m_{jkl} B_{abm} \Upsilon_c^l + \Xi \eta^{abc} \epsilon_{plj} B_{ab}^{0p} A_{c0}^l \right] \\ &\quad + \epsilon^i_{kj} A_{00i} V^k \left[\Xi \eta^{abc} \partial_c B_{ab}^{0j} - \Xi \eta^{abc} \epsilon^j_i{}^m B_{abm} A_{c0}^l - \Xi \eta \epsilon^j_{nl} B_{ab}^{0n} \Upsilon_c^l \right] \\ &\quad - \epsilon^i_{jk} B_{0ai} V^k \left[\Xi \eta^{abc} \left(\partial_b \Upsilon_c^j - \partial_c \Upsilon_b^j \right) + \Xi \eta^{abc} \epsilon^j_{pl} \Upsilon_b^p \Upsilon_c^l - \Xi \eta^{abc} \epsilon^j_{pl} A_{b0}^p A_{c0}^l \right] \\ &\quad - \epsilon_{ijk} B_{0a}^{0i} V^k \left[\Xi \eta^{abc} \left(\partial_b A_{c0}^j - \partial_c A_{b0}^j \right) + \epsilon^j_{ml} \Upsilon_c^l A_{b0}^m - \epsilon_{jml} \Upsilon_b^l A_{c0}^m \right] \\ &= 0, \end{aligned} \tag{21}$$

where we can observe that the left hand side vanishes because is a linear combination of constraints. Hence, there are not more FJ constraints.

Furthermore, we will add the constraints given in (13) to the symplectic Lagrangian using the

following Lagrange multipliers, namely, $A_0^i = \dot{T}^i$, $A_{00i} = \dot{\Lambda}_i$, $B_{0a}^{0i} = \frac{\dot{\varsigma}_a^i}{2}$, $B_{0ai} = \frac{\dot{\chi}_{ai}}{2}$, thus the symplectic Lagrangian takes the form

$$\begin{aligned} \mathcal{L}^{(1)} = & \Xi\eta^{abc}B_{ab}^{0i}\dot{A}_{c0i} + \Xi\eta^{abc}B_{abi}\dot{\Upsilon}_c^i - \dot{T}^i\Omega^{(0)}_i - \dot{\Lambda}_i\Omega^{(0)}_{00i} - \frac{\dot{\varsigma}_a^i}{2}\Omega^{(0)}_{0ai} \\ & - \frac{\dot{\chi}_{ai}}{2}\Omega^{(0)}_{0ai} - \mathcal{V}^{(1)}, \end{aligned} \quad (22)$$

where $\mathcal{V}^{(1)} = \mathcal{V}^{(0)}|_{\Omega^{(0)}_i, \Omega^{(0)}_{00i}, \Omega^{(0)}_{0a0i}, \Omega^{(0)}_{0ai}=0} = 0$, this result is expected because of the general covariance of the theory just as it is present in General Relativity.

From the symplectic Lagrangian (22) we identify the following symplectic variables $\xi^{(1)} = (A_{c0i}, B_{ab}^{0i}, \Upsilon_c^i, B_{abi}, T^i, \Lambda_i, \varsigma_a^i, \chi_{ai})$ and the 1-forms $a^{(1)} = \left(\Xi\eta^{abc}B_{ab}^{0i}, 0, \Xi\eta^{abc}B_{abi}, 0, -\Omega^{(0)}_i, -\Omega^{(0)}_{00i}, -\frac{\Omega^{(0)}_{0a0i}}{2}, -\frac{\Omega^{(0)}_{0ai}}{2} \right)$. Hence, the symplectic matrix has the following form

$$f^{(1)}_{ij} = \begin{pmatrix} 0 & -\Xi\eta^{abc}\delta_j^i & 0 & 0 \\ \Xi\eta^{abc}\delta_j^i & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Xi\eta^{abc}\delta_j^i \\ 0 & 0 & \Xi\eta^{abc}\delta_j^i & 0 \\ -\Xi\eta^{abc}\epsilon_j^k B_{ab}^{0j} & -\Xi\eta^{abc}\epsilon_{kji}A_{c0}^j & \Xi\eta^{abc}\epsilon_{ik}^j B_{abj} & \Xi\eta^{abc}D_c^k{}_i \\ -\Xi\eta^{abc}\epsilon^{ikj}B_{abj} & \Xi\eta^{abc}d_c^i{}_k & \Xi\eta^{abc}\epsilon_{jk}^i B_{ab}^{0i} & \Xi\eta^{abc}\epsilon^{ijk}A_{c0j} \\ \Xi\eta^{abc}d_{bi}^k & 0 & \Xi\eta^{abc}\epsilon_i^j{}_k A_{b0j} & 0 \\ -\Xi\eta^{abc}\epsilon^{ijk}A_{b0j} & 0 & \Xi\eta^{abc}d_b^i{}_k & 0 \\ \Xi\eta^{abc}\epsilon_j^k{}_i B_{ab}^{0j} & \Xi\eta^{abc}\epsilon^{ikj}B_{abj} & \Xi\eta^{abc}d_{ai}^k & \Xi\eta^{abc}\epsilon^{ijk}A_{b0j} \\ \Xi\eta^{abc}\epsilon_{kji}A_{c0}^j & -\Xi\eta^{abc}d_c^i{}_k & 0 & 0 \\ -\Xi\eta^{abc}\epsilon_{ik}^j B_{abj} & \Xi\eta^{abc}\epsilon_{jk}^i B_{ab}^{0i} & -\Xi\eta^{abc}\epsilon_{ik}^j A_{b0j} & -\Xi\eta^{abc}D_b^i{}_j \\ -\Xi\eta^{abc}D_c^k{}_i & \Xi\eta^{abc}\epsilon^{ijk}A_{c0j} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \delta^3(x-y), \quad (23)$$

where we have used the notation $D_{al}^i = \delta_l^i \partial_a + \epsilon_l^{ik} \Upsilon_{ak}$ and $d_{ai}^l = \delta_i^l \partial_a - \epsilon_i^{lk} \Upsilon_{ak}$. We can observe that this matrix is singular, however we have showed that there are not more constraints, therefore the theory under study has a gauge symmetry. In order to obtain a symplectic tensor, we fix the following temporal gauge

$$A_0^i = 0, \quad (24)$$

$$A_{00i} = 0, \quad (25)$$

$$B_{0a}^{0i} = 0, \quad (26)$$

$$B_{0ai} = 0, \quad (27)$$

this mean that $\dot{T}^i = 0$, $\dot{\Lambda}_i = 0$, $\dot{\varsigma}_a^i = 0$ and $\dot{\chi}_{ai} = 0$. In this manner, we introduce more Lagrange

multipliers enforcing the gauge fixing. The Lagrange multipliers introduced are $\beta_i, \alpha^i, \rho_i^a, \sigma_a^i$, thus, the symplectic Lagrangian takes the form

$$\begin{aligned} \mathcal{L}^{(2)} = & \Xi\eta^{abc} B_{ab}{}^{0i} \dot{A}_{c0i} + \Xi\eta^{abc} B_{abi} \dot{\Upsilon}_c{}^i - \left[\Omega^{(0)}{}_i - \beta_i \right] \dot{T}^i - \left[\Omega^{(0)}{}^{00i} - \alpha^i \right] \dot{\Lambda}_i \\ & - \left[\frac{\Omega^{(0)}{}_{0a}{}^{0i}}{2} - \rho_i^a \right] \dot{\varsigma}_a^i - \left[\frac{\Omega^{(0)}{}_{0ai}}{2} - \sigma^{ai} \right] \dot{\chi}_{ai}, \end{aligned} \quad (28)$$

from this symplectic Lagrangian we identify the following symplectic variables $\xi^{(2)} = (A_{a0i}, B_{ab}{}^{0i}, \Upsilon_a^i, B_{abi}, T^i, \Lambda_i, \varsigma_a^i, \chi_{ai}, \beta_i, \alpha^i, \rho_i^a, \sigma^{ai})$, and the 1-forms

$$\begin{aligned} a^{(2)} = & \left(\Xi\eta^{abc} B_{ab}{}^{0i}, 0, \Xi\eta^{abc} B_{abi}, 0, -\left[\Omega^{(0)}{}_i - \beta_i \right], -\left[\Omega^{(0)}{}^{00i} - \alpha_i \right], \right. \\ & \left. -\left[\frac{\Omega^{(0)}{}_{0a}{}^{0i}}{2} - \rho_i \right], -\left[\frac{\Omega^{(0)}{}_{0ai}}{2} - \sigma_i \right], 0, 0, 0, 0 \right). \end{aligned}$$

Thus, the symplectic matrix is given by

$$\begin{aligned} f^{(2)}_{ij} = & \begin{pmatrix} 0 & -\Xi\eta^{abc}\delta_j^i & 0 & 0 & \Xi\eta^{abc}\epsilon_j{}^i{}_k B_{ab}{}^{0j} \\ \Xi\eta^{abc}\delta_j^i & 0 & 0 & 0 & \Xi\eta^{abc}\epsilon_{ijk} A_{c0}{}^j \\ 0 & 0 & 0 & -\Xi\eta^{abc}\delta_j^i & -\Xi\eta^{abc}\epsilon^j{}_{ki} B_{abj} \\ 0 & 0 & \Xi\eta^{abc}\delta_j^i & 0 & -\Xi\eta^{abc}\epsilon^i{}_{jk} \Upsilon_c^k \\ -\Xi\eta^{abc}\epsilon_j{}^i{}_k B_{ab}{}^{0j} & -\Xi\eta^{abc}\epsilon_{ijk} A_{c0}{}^j & \Xi\eta^{abc}\epsilon^j{}_{ki} B_{abj} & \Xi\eta^{abc}\epsilon^i{}_{jk} \Upsilon_c^k & 0 \\ -\Xi\eta^{abc}\epsilon^{kij} B_{abj} & \Xi\eta^{abc} d_c{}^i{}_l & -\Xi\eta^{abc}\epsilon^k{}_{ji} B_{ab}{}^{0j} & -\Xi\eta^{abc}\epsilon^{kji} A_{c0j} & 0 \\ -\Xi\eta^{abc} d_{bl}{}^i & 0 & -\Xi\eta^{abc}\epsilon_i{}^{jk} A_{b0j} & 0 & 0 \\ -\Xi\eta^{abc}\epsilon^{kji} A_{b0j} & 0 & \Xi\eta^{abc} d_b{}^l{}_i & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_j^i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ & \begin{pmatrix} \Xi\eta^{abc}\epsilon^{kij} B_{abj} & \Xi\eta^{abc} d_{bj}{}^i & \Xi\eta^{abc}\epsilon^{kji} A_{b0j} & 0 & 0 & 0 & 0 \\ -\Xi\eta^{abc} d_c{}^i{}_l & 0 & 0 & 0 & 0 & 0 & 0 \\ \Xi\epsilon^k{}_{ji} B_{ab}{}^{0j} & \Xi\eta^{abc}\epsilon_i{}^{jk} A_{b0j} & -\Xi\eta^{abc} d_b{}^l{}_i & 0 & 0 & 0 & 0 \\ \Xi\eta^{abc}\epsilon^{kji} A_{c0j} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta_j^i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta_j^i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\delta_b^a \delta_j^i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\delta_b^a \delta_j^i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta_j^i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta_b^a \delta_j^i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \delta_b^a \delta_j^i & 0 & 0 & 0 & 0 \end{pmatrix} \delta^3(x-y), \quad (29) \end{aligned}$$

where $d_{ai}^l \equiv \delta_i^l \partial_a - \epsilon_i^{lk} \Upsilon_{ak}$. We can observe that this matrix is not singular, after a long calculation, the inverse of $f_{ij}^{(2)}$ is given by

$$f_{ij}^{(2)-1} = \begin{pmatrix} 0 & \frac{1}{2\Xi} \eta_{bgc} \delta_j^i & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2\Xi} \eta_{bgc} \delta_j^i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\Xi} \eta_{abc} \delta_j^i & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2\Xi} \eta_{abc} \delta_j^i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \epsilon_{ijl} A_{c0}^j & -\epsilon_j^l B_{ab}^{0j} & -\epsilon_{jk}^l \Upsilon_c^k & -\epsilon_{kl}^j B_{mnj} & -\delta_j^i & 0 & 0 & 0 \\ 0 & -\epsilon^{klj} B_{abj} & \epsilon^{kjl} A_{c0j} & -\frac{1}{2} \delta_{mn}^{ab} \epsilon_k^j A_{b0j} & 0 & -\delta_j^i & 0 & 0 \\ 0 & -\frac{1}{2} \delta_{mn}^{ab} D_{bi}^l & 0 & -\frac{1}{2} \delta_{mn}^{ab} D_b^k & 0 & 0 & -\delta_b^a \delta_j^i & 0 \\ 0 & -\frac{1}{2} \delta_{mn}^{ab} \epsilon^{kjl} A_{b0j} & 0 & 0 & 0 & 0 & 0 & -\delta_b^a \delta_j^i \\ -\epsilon_{ijl} A_{c0}^j & 0 & 0 & 0 & & & & \\ \epsilon_j^l B_{ab}^{0j} & \epsilon^{klj} B_{abj} & \frac{1}{2} \delta_{mn}^{ab} D_{ai}^l & \frac{1}{2} \delta_{mn}^{ab} \epsilon^{kjl} A_{b0j} & & & & \\ \epsilon_{jk}^l \Upsilon_c^k & -\epsilon^{kjl} A_{c0j} & 0 & 0 & & & & \\ \epsilon_{kl}^j B_{mnj} & \frac{1}{2} \delta_{mn}^{ab} \epsilon_k^j A_{b0j} & \frac{1}{2} \delta_{mn}^{ab} D_a^k & 0 & & & & \\ \delta_j^i & 0 & 0 & 0 & & & & \\ 0 & \delta_j^i & 0 & 0 & & & & \\ 0 & 0 & \delta_b^a \delta_j^i & 0 & & & & \\ 0 & 0 & 0 & \delta_b^a \delta_j^i & & & & \\ 0 & \frac{\Xi}{2} H^{mi}{}_{j}{}^q{}^l & -\frac{\Xi}{2} E^{ai}{}_j & \frac{\Xi}{2} F^m{}_i{}^l & & & & \\ -\frac{\Xi}{2} H^{mi}{}_{j}{}^q{}^l & \Xi \eta^{abc} \epsilon^{ijk} B_{abk} A_{c0j} & 0 & G^{mjpi} & & & & \\ \frac{\Xi}{2} E^{ai}{}_j & 0 & 0 & 0 & & & & \\ \frac{\Xi}{2} F^m{}_i{}^l & -G^{kjpi} & 0 & 0 & & & & \end{pmatrix} \delta^3(x-y), \quad (30)$$

where we have defined

$$\begin{aligned} D_{al}^i &\equiv \delta_l^i \partial_a + \epsilon_l^{ik} \Upsilon_{ak}, \\ E^{ai}{}_j &\equiv \eta^{abc} \epsilon_{jp}^i \epsilon^{plk} A_{b0l} A_{c0k} + \eta^{abc} \epsilon_{pj}^l \epsilon_{kl}^i \Upsilon_b^k \Upsilon_c^p, \\ F^m{}_i{}^l &\equiv \eta^{mnc} \epsilon_{ijk} \epsilon^{kpl} A_{c0}^j A_{n0p}, \\ G^{mjpi} &\equiv \eta^{mnc} \epsilon_{jk}^j \epsilon_{li}^p A_{n0l} \Upsilon_c^k, \\ H^{mi}{}_{j}{}^q{}^l &\equiv \eta^{abc} \delta_{ab}^{mn} \epsilon_{jk}^i \epsilon_{li}^{qp} A_{n0p} \Upsilon_c^k. \end{aligned}$$

Therefore, from the symplectic tensor (30) we can identify the generalized FJ brackets by means of

$$\{\xi_i^{(2)}(x), \xi_j^{(2)}(y)\}_{FD} = [f_{ij}^{(2)}(x, y)]^{-1}, \quad (31)$$

thus, the following generalized brackets arise

$$\left\{A_{c0i}(x), B_{ab}^{0j}(y)\right\}_{FJ} = \frac{1}{2\Xi} \eta_{abc} \delta_i^j \delta^3(x-y), \quad (32)$$

$$\left\{\Upsilon_c^i(x), B_{abj}(y)\right\}_{FJ} = \frac{1}{2\Xi} \eta_{abc} \delta_j^i \delta^3(x-y), \quad (33)$$

where we can observe that the FJ brackets and the Dirac ones coincide to each other (see the section below). Furthermore, in FJ framework there are less constraints than in Dirac's framework, in this sense, the FJ is more economical to perform; we will see this point in more details the following section. Finally, we carry out the counting of physical degrees of freedom. As we have commented above, in FJ formalism there are not a classification of constraints, they are at the same level, thus, the counting of physical degrees of freedom is performed as $[DF = \text{dynamical variables} - \text{independent constraints}]$. In this manner, there are 18 canonical variables given by (A_{coi}, Υ_a^i) and 18 independent first class constraints $(\Omega^{(0)}_i, \Omega^{(0)00i}, \Omega^{(0)0ai}, \Omega^{(0)0a}_{0i})$; for BF theory it is well-knew that the constraints are reducible, the reducibility between the constraints is given by $\partial_a \Omega^{(0)0ai} = \epsilon_{ij}^k \Upsilon_{ak} \Omega^{(0)0aj} + \epsilon_{ij}^k A_{a0k} \Omega^{(0)0a}_{0j}$ and $\partial_a \Omega^{(0)0a}_{0i} = \epsilon_i^{jk} \Upsilon_{ak} \Omega^{(0)0a}_{0j} + \epsilon_i^k A_{a0k} \Omega^{(0)0aj}$. Therefore, the theory is devoid of physical degrees of freedom as expected. It is important to comment, that all results found in this section are not reported in the literature.

III. HAMILTONIAN ANALYSIS

In this section a pure Dirac's canonical analysis for the four-dimensional BF theory will be performed, we will follow all Dirac's steps in order to obtain the better canonical description of the theory [16]. For this aim, we start with the Lagrangian given in (2)

$$\begin{aligned} L = \Xi \int \eta^{abc} \Big[& B_{ab}^{0i} \dot{A}_{c0i} + \frac{1}{2} B_{ab}^{ij} \dot{A}_{cij} \\ & + \frac{1}{2} A_{0ij} \left(\partial_c B_{ab}^{ij} + B_{ab}^{il} A_c^j{}_l + 2 B_{ab}^{0i} A_{c0}^j \right) \\ & + A_{00i} \left(\partial_c B_{ab}^{0i} + B_{ab}^{0j} A_c^i{}_j + B_{ab}^{ij} A_{c0j} \right) \\ & + B_{0a}^{0i} \left(\partial_b A_{c0i} - \partial_c A_{b0i} + A_{b0j} A_c^j{}_i + A_{c0}^j A_{bij} \right) \\ & + \frac{1}{2} B_{0a}^{ij} \left(\partial_b A_{cij} - \partial_c A_{bij} + A_{bil} A_c^l{}_j + A_{bi0} A_c^0{}_j - A_{bjl} A_c^l{}_i - A_{bj0} A_c^0{}_i \right) \Big] d^3x, \end{aligned} \quad (34)$$

by considering the following change of variables [28, 30]

$$\begin{aligned}
A_{aij} &\equiv -\epsilon_{ijk} A_a^k, \\
A_{0ij} &\equiv -\epsilon_{ijk} A_0^k, \\
B_{abij} &\equiv -\epsilon_{ijk} B_{ab}^k, \\
B_{0aij} &\equiv -\epsilon_{ijk} B_{0a}^k, \\
A_{a0i} &\equiv A_{a0i}, \\
A_{ai} &\equiv \Upsilon_{ai}, \\
A_0^i &\equiv -T^i, \\
A_{00i} &\equiv -\Lambda_i, \\
B_{0a}^{0i} &\equiv -\frac{1}{2} \varsigma_a^i, \\
B_{0ai} &\equiv -\frac{1}{2} \chi_{ai},
\end{aligned} \tag{35}$$

the Lagrangian takes the following form

$$\begin{aligned}
L &= L[A_{a0i}, \Upsilon_{ai}, T_i, \Lambda_i, \varsigma_{ai}, \chi_{ai}, B_{ab0i}, B_{abi}] \\
&= \int \left[\Xi \eta^{abc} B_{ab}^{0i} \dot{A}_{c0i} + \Xi \eta^{abc} B_{abi} \dot{\Upsilon}_c^i \right. \\
&\quad - T^i \left(\partial_c (\Xi \eta^{abc} B_{abi}) + \Xi \eta^{abc} \epsilon_{ik}^j B_{abj} \Upsilon_c^k - \Xi \eta^{abc} \epsilon_{jki} B_{ab}^{0j} A_{c0}^k \right) \\
&\quad - \Lambda_i \left(\partial_c (\Xi \eta^{abc} B_{ab}^{0i}) - \Xi \eta^{abc} \epsilon_{jk}^i B_{ab}^{0j} \Upsilon_c^k - \Xi \eta^{abc} \epsilon^{ijk} B_{abk} A_{c0j} \right) \\
&\quad - \frac{1}{2} \Xi \eta^{abc} \varsigma_a^i \left(\partial_b A_{c0i} - \partial_c A_{b0i} + \epsilon_i^{jk} A_{b0j} \Upsilon_{ck} - \epsilon_{ijk} A_{c0}^j \Upsilon_b^k \right) \\
&\quad \left. - \frac{1}{2} \Xi \eta^{abc} \chi_{ai} \left(\partial_b \Upsilon_c^i - \partial_c \Upsilon_b^i + \epsilon^i_{jk} \Upsilon_b^j \Upsilon_c^k - \epsilon^{ijk} A_{b0j} A_{c0k} \right) \right] d^3x.
\end{aligned} \tag{36}$$

In this manner, the canonically momenta of the dynamical variables are given by

$$\begin{aligned}
p^{a0i} &\equiv \frac{\partial L}{\partial \dot{A}_{a0i}} = \Xi \eta^{abc} B_{bc}^{0i}, \\
\pi^{ai} &\equiv \frac{\partial L}{\partial \dot{\Upsilon}_{ai}} = \Xi \eta^{abc} B_{bc}^i, \\
\hat{T}^i &\equiv \frac{\partial L}{\partial \dot{T}_i} = 0, \\
\hat{\Lambda}^i &\equiv \frac{\partial L}{\partial \dot{\Lambda}_i} = 0, \\
\varsigma^{ai} &\equiv \frac{\partial L}{\partial \dot{\varsigma}_{ai}} = 0, \\
\hat{\chi}^{ai} &\equiv \frac{\partial L}{\partial \dot{\chi}_{ai}} = 0, \\
p^{ab0i} &\equiv \frac{\partial L}{\partial \dot{B}_{ab0i}} = 0, \\
p^{abi} &\equiv \frac{\partial L}{\partial \dot{B}_{abi}} = 0,
\end{aligned} \tag{37}$$

with the following non-vanishing fundamental Poisson brackets between the fields

$$\begin{aligned}
\{\Upsilon_{ai}(x), \pi^{bj}(y)\} &= \frac{1}{2} \delta_a^b \delta_i^j \delta^3(x-y), \\
\{A_{a0i}(x), p^{b0j}(y)\} &= \frac{1}{2} \delta_a^b \delta_i^j \delta^3(x-y), \\
\{T_i(x), \hat{T}^j(y)\} &= \frac{1}{2} \delta_i^j \delta^3(x-y), \\
\{\Lambda_i(x), \hat{\Lambda}^j(y)\} &= \frac{1}{2} \delta_i^j \delta^3(x-y), \\
\{\varsigma_{ai}(x), \varsigma^{bj}(y)\} &= \delta_a^b \delta_i^j \delta^3(x-y), \\
\{\chi_{ai}(x), \hat{\chi}^{bj}(y)\} &= \delta_a^b \delta_i^j \delta^3(x-y), \\
\{B_{ab0i}(x), p^{de0j}(y)\} &= \frac{1}{4} (\delta_a^d \delta_b^e - \delta_a^e \delta_b^d) \delta_i^j \delta^3(x-y), \\
\{B_{abi}(x), p^{dej}(y)\} &= \frac{1}{4} (\delta_a^d \delta_b^e - \delta_a^e \delta_b^d) \delta_i^j \delta^3(x-y).
\end{aligned} \tag{38}$$

Furthermore, from the definition of the momenta, we identify the following 60 primary constraints

$$\begin{aligned}
\phi_1^{a0i} &\equiv p^{a0i} - \Xi \eta^{abc} B_{bc}{}^{0i} \approx 0, \\
\phi_2^{ai} &\equiv \pi^{ai} - \Xi \eta^{abc} B_{bc}{}^i \approx 0, \\
\phi_3^i &\equiv \hat{T}^i \approx 0, \\
\phi_4^i &\equiv \hat{\Lambda}^i \approx 0, \\
\phi_5^{ai} &\equiv \varsigma^{ai} \approx 0, \\
\phi_6^{ai} &\equiv \hat{\chi}^{ai} \approx 0, \\
\phi_7^{ab0i} &\equiv p^{ab0i} \approx 0, \\
\phi_8^{abi} &\equiv p^{abi} \approx 0.
\end{aligned} \tag{39}$$

The canonical Hamiltonian of the theory is given by

$$\begin{aligned}
H_c &= \int \left[\dot{A}_{a0i} p^{a0i} + \dot{\Upsilon}_{ai} \pi^{ai} + \dot{T}_i \hat{T}^i + \dot{\Lambda}_i \hat{\Lambda}^i + \dot{\varsigma}_{ai} \varsigma^{ai} + \dot{B}_{ab0i} p^{ab0i} + \dot{B}_{abi} p^{abi} - L \right] d^3x \\
&= \int \left[T^i \left(\partial_a \pi^a{}_i - \epsilon_i{}^{jk} \pi^a{}_j \Upsilon_{ak} - \epsilon_{ijk} p^{a0j} A_{a0}{}^k \right) \right. \\
&\quad + \Lambda_i \left(\partial_a p^{a0i} - \epsilon^i{}_{jk} p^{a0j} \Upsilon_a{}^k - \epsilon^{ijk} \pi^a{}_k A_{a0j} \right) \\
&\quad + \frac{1}{2} \Xi \eta^{abc} \varsigma_a{}^i \left(\partial_b A_{c0i} - \partial_c A_{b0i} + \epsilon_i{}^{jk} A_{b0j} \Upsilon_{ck} + \epsilon_{ijk} A_{c0}{}^j \Upsilon_b{}^k \right) \\
&\quad \left. + \frac{1}{2} \Xi \eta^{abc} \chi_{ai} \left(\partial_b \Upsilon_c{}^i - \partial_c \Upsilon_b{}^i + \epsilon^i{}_{jk} \Upsilon_b{}^j \Upsilon_c{}^k - \epsilon^{ijk} A_{b0j} A_{c0k} \right) \right] d^3x,
\end{aligned} \tag{40}$$

by adding the primary constraints we obtain the primary Hamiltonian

$$\begin{aligned}
H_1 &= \int \left[H_c + T^i \left(\partial_a \pi^a{}_i - \epsilon_i{}^{jk} \pi^a{}_j \Upsilon_{ak} - \epsilon_{ijk} p^{a0j} A_{a0}{}^k \right) \right. \\
&\quad + \Lambda_i \left(\partial_a p^{a0i} - \epsilon^i{}_{jk} p^{a0j} \Upsilon_a{}^k - \epsilon^{ijk} \pi^a{}_k A_{a0j} \right) \\
&\quad + \frac{1}{2} \Xi \eta^{abc} \varsigma_a{}^i \left(\partial_b A_{c0i} - \partial_c A_{b0i} + \epsilon_i{}^{jk} A_{b0j} \Upsilon_{ck} + \epsilon_{ijk} A_{c0}{}^j \Upsilon_b{}^k \right) \\
&\quad + \frac{1}{2} \Xi \eta^{abc} \chi_{ai} \left(\partial_b \Upsilon_c{}^i - \partial_c \Upsilon_b{}^i + \epsilon^i{}_{jk} \Upsilon_b{}^j \Upsilon_c{}^k - \epsilon^{ijk} A_{b0j} A_{c0k} \right) \\
&\quad + \lambda_{a0i} \left(p^{a0i} - \Xi \eta^{abc} B_{bc}{}^{0i} \right) + \lambda_{ai} \left(\pi^{ai} - \Xi \eta^{abc} B_{bc}{}^i \right) + \alpha_i \hat{T}^i + \beta_i \hat{\Lambda}^i \\
&\quad \left. + \theta_{ai} \varsigma^{ai} + \mu_{ai} \hat{\chi}^{ai} + \lambda_{ab0i} p^{ab0i} + \lambda_{abi} p^{abi} \right] d^3x,
\end{aligned} \tag{41}$$

where $\lambda_{a0i}, \lambda_{ai}, \alpha_i, \beta_i, \theta_{ai}, \mu_{ai}, \lambda_{ab0i}, \lambda_{abi}$ are Lagrange multipliers enforcing the primary constraints.

From consistency of the constraints, we identify the following 24 secondary constraints

$$\dot{\phi}_3^i \approx 0 \Rightarrow \varphi_3^i \equiv -\frac{1}{2} [\partial_a \pi^{ai} - \epsilon^{ijk} \pi^a_j \Upsilon_{ak} - \epsilon^i_{jk} p^{a0j} A_{a0}^k] \approx 0, \quad (42)$$

$$\dot{\phi}_4^i \approx 0 \Rightarrow \varphi_4^i \equiv -\frac{1}{2} [\partial_a p^{a0i} - \epsilon^i_{jk} p^{a0j} \Upsilon_a^k - \epsilon^{ijk} \pi^a_k A_{a0j}] \approx 0, \quad (43)$$

$$\begin{aligned} \dot{\phi}_5^{ai} \approx 0 \Rightarrow \varphi_5^{ai} &\equiv -\frac{\Xi}{2} \eta^{abc} (\partial_b A_c^{0i} - \partial_c A_b^{0i} - \epsilon^{ijk} A_{b0j} \Upsilon_{ck} + \epsilon^{ijk} A_{c0j} \Upsilon_{bk}) \\ &\approx 0, \end{aligned} \quad (44)$$

$$\begin{aligned} \dot{\phi}_6^{ai} \approx 0 \Rightarrow \varphi_6^{ai} &\equiv -\frac{\Xi}{2} \eta^{abc} [\partial_b \Upsilon_c^i - \partial_c \Upsilon_b^i + \epsilon^{ijk} \Upsilon_{bj} \Upsilon_{ck} - \epsilon^{ijk} A_{b0j} A_{c0k}] \\ &\approx 0, \end{aligned} \quad (45)$$

and the following 36 Lagrange multipliers

$$\begin{aligned} \lambda_{ab}^{0i} &\approx \frac{1}{2\Xi} (\eta_{abc} \epsilon^i_{jk} T^j p^{c0k} - \eta_{abc} \epsilon^{ijk} \Lambda_j \pi^c_k + \Xi \partial_b \varsigma_a^i - \Xi \partial_a \varsigma_b^i - \Xi \epsilon^i_j{}^k \varsigma_a^j \Upsilon_{bk} \\ &\quad + \Xi \epsilon^i_j{}^k \varsigma_b^j \Upsilon_{ak} + \epsilon^{ijk} \chi_{aj} A_{b0k} - \epsilon^{ijk} \chi_{bj} A_{a0k}), \end{aligned} \quad (46)$$

$$\begin{aligned} \lambda_{ab}^i &\approx \frac{1}{2\Xi} (\eta_{abc} \epsilon^i_j{}^k T^j \pi^c_k + \eta_{abc} \epsilon^{ij}{}_k p^{c0k} \Lambda_j - \Xi \epsilon^i_j{}^k \varsigma_a^j A_{b0k} + \Xi \epsilon^i_j{}^k \varsigma_b^j A_{a0k} \\ &\quad + \Xi \partial_b \chi_a^i - \Xi \partial_a \chi_b^i - \Xi \epsilon^{ijk} \chi_{aj} \Upsilon_{bk} + \Xi \epsilon^{ijk} \chi_{bj} \Upsilon_{ak}), \end{aligned} \quad (47)$$

$$\lambda_{a0i} \approx 0, \quad (48)$$

$$\lambda_{ai} \approx 0. \quad (49)$$

For this theory there are not tertiary constraints. Hence, in order to perform the classification of the constraints in first class and second class we proceed to calculate the following matrix whose

entries are the Poisson brackets between all the constraints, it is

$$\{\varphi_3^i(x), \varphi_3^l(y)\} = -\frac{1}{4}\epsilon^{ilk}\varphi_{3k}\delta^3(x-y) \approx 0, \quad (50)$$

$$\{\varphi_4^i(x), \varphi_4^l(y)\} = \frac{1}{4}\epsilon^{ilk}\varphi_{3k}\delta^3(x-y) \approx 0. \quad (51)$$

$$\{\phi_1^{a0i}(x), \phi_7^{de0l}(y)\} = \frac{\Xi}{2}\eta^{ade}\eta^{il}\delta^3(x-y), \quad (52)$$

$$\{\phi_1^{a0i}(x), \varphi_3^l(y)\} = -\frac{1}{4}\epsilon^{il}{}_jp^{a0j}\delta^3(x-y), \quad (53)$$

$$\{\phi_1^{a0i}(x), \varphi_4^l(y)\} = \frac{1}{4}\epsilon^{ilj}\pi^a{}_j\delta^3(x-y), \quad (54)$$

$$\{\phi_1^{a0i}(x), \varphi_5^{dl}(y)\} = -\frac{\Xi}{2}[-\eta^{il}\eta^{ade}\partial_{x,e} + \eta^{ade}\epsilon^{ilj}\Upsilon_{ej}]\delta^3(x-y), \quad (55)$$

$$\{\phi_1^{a0i}(x), \varphi_6^{dl}(y)\} = -\frac{\Xi}{2}\eta^{ade}\epsilon^{ilj}A_{e0j}\delta^3(x-y), \quad (56)$$

$$\{\phi_2^{ai}(x), \phi_8^{del}(y)\} = -\frac{\Xi}{2}\eta^{il}\eta^{ade}\delta^3(x-y), \quad (57)$$

$$\{\phi_2^{ai}(x), \varphi_3^l(y)\} = -\frac{1}{4}\epsilon^{ilj}\pi^a{}_j\delta^3(x-y), \quad (58)$$

$$\{\phi_2^{ai}(x), \varphi_4^l(y)\} = -\frac{1}{4}\epsilon^{il}{}_jp^{a0j}\delta^3(x-y), \quad (59)$$

$$\{\phi_2^{ai}(x), \varphi_5^{dl}(y)\} = -\frac{\Xi}{2}\eta^{ade}\epsilon^{ilj}A_{e0j}\delta^3(x-y), \quad (60)$$

$$\{\phi_2^{ai}(x), \varphi_6^{dl}(y)\} = \frac{\Xi}{2}\eta^{ade}[-\eta^{il}\partial_{x,e} + \epsilon^{ilj}\Upsilon_{ej}]\delta^3(x-y), \quad (61)$$

$$\{\varphi_3^i(x), \varphi_4^l(y)\} = -\frac{1}{4}\epsilon^{il}{}_j\varphi_4^j\delta^3(x-y) \approx 0, \quad (62)$$

$$\{\varphi_3^i(x), \varphi_5^{dl}(y)\} = -\frac{1}{4}\epsilon^{il}{}_k\varphi_5^{dk}\delta^3(x-y) \approx 0, \quad (63)$$

$$\{\varphi_3^i(x), \varphi_6^{dl}(y)\} = -\frac{1}{4}\epsilon^{il}{}_k\varphi_6^{dk}\delta^3(x-y) \approx 0, \quad (64)$$

$$\{\varphi_4^i(x), \varphi_5^{dl}(y)\} = \frac{1}{4}\epsilon^{il}{}_k\varphi_6^k\delta^3(x-y) \approx 0, \quad (65)$$

$$\{\varphi_4^i(x), \varphi_6^{dl}(y)\} = -\frac{1}{4}\epsilon^{il}{}_j\varphi_5^{dj}\delta^3(x-y) \approx 0. \quad (66)$$

we find that this matrix has rank = 36 and 48 null vectors. From the null vectors we find the following 48 first class constraints

$$\begin{aligned}
\gamma_1^i &\equiv \hat{T}^i \approx 0, \\
\gamma_2^i &\equiv \hat{\Lambda}^i \approx 0, \\
\gamma_3^{ai} &\equiv \hat{\zeta}^{ai} \approx 0, \\
\gamma_4^{ai} &\equiv \hat{\chi}^{ai} \approx 0, \\
\gamma_{5\ i} &\equiv \partial_a \pi^a_i - \epsilon_i^{jk} \pi^a_j \Upsilon_{ak} - \epsilon_{ijk} p^{a0j} A_{a0}^k \\
&\quad + \frac{1}{2\Xi} \eta_{abc} \epsilon_{ijk} (\pi^{aj} p^{bck} - p^{a0j} p^{bc0k}) \approx 0, \\
\gamma_6^{0i} &\equiv \partial_a p^{a0i} - \epsilon^i_{jk} p^{a0j} \Upsilon_a^k - \epsilon^{ijk} \pi^a_k A_{a0j} \\
&\quad + \frac{1}{2\Xi} \eta_{abc} \epsilon^i_{jk} (\pi^{aj} p^{bc0k} + p^{a0j} p^{bck}) \approx 0, \\
\gamma_7^a{}_{0i} &\equiv \frac{\Xi}{2} \eta^{abc} \left(\partial_b A_{c0i} - \partial_c A_{b0i} + \epsilon_i^{jk} A_{b0j} \Upsilon_{ck} - \epsilon_{ijk} A_{c0}^j \Upsilon_b^k \right) \\
&\quad - \partial_b p^{ab}{}_{0i} + \epsilon_i^{jk} \Upsilon_{bk} p^{ab}{}_{0j} + \epsilon_i^{jk} A_{b0k} p^{ab}{}_j \approx 0, \\
\gamma_8^{ai} &\equiv \frac{\Xi}{2} \eta^{abc} [\partial_b \Upsilon_c^i - \partial_c \Upsilon_b^i + \epsilon^{ijk} \Upsilon_{bj} \Upsilon_{ck} - \epsilon^{ijk} A_{b0j} A_{c0k}] \\
&\quad - \partial_b p^{abi} - \epsilon^{ijk} A_{b0k} p^{ab}{}_{0j} + \epsilon^{ijk} \Upsilon_{bk} p^{ab}{}_j \approx 0.
\end{aligned} \tag{67}$$

and the rank allows us identify the following 36 second class constraints

$$\begin{aligned}
\Gamma_1^{a0i} &\equiv p^{a0i} - \Xi \eta^{abc} B_{bc}{}^{0i} \approx 0, \\
\Gamma_2^{ai} &\equiv \pi^{ai} - \Xi \eta^{abc} B_{bc}{}^i \approx 0, \\
\Gamma_3^{ab0i} &\equiv p^{ab0i} \approx 0, \\
\Gamma_4^{abi} &\equiv p^{abi} \approx 0.
\end{aligned} \tag{68}$$

It is important to remark that the complete structure of the constraints (67) is not reported in the literature and this is a result of performing a pure Dirac's formulation. In fact, by working with the standard form, it is not possible to obtain a full structure of the constraints and the algebra could not be closed just as is present in four-dimensional Palatini's theory [15]. Furthermore, in order to compare the symplectic framework with the Dirac one, it is necessary to follow all steps of the Dirac formulation as has been developed in this paper.

With all information obtained, we can carryout the counting of physical degrees of the theory in the following form; there are 120 dynamical variables, 48 first class constraints and 36 second class constraints, hence we obtain -6 degrees of freedom. However, it is well-known that BF theory is a reducible theory, this is, the constraints are not independent to each other. The reducibility of the constraints are given by

$$\begin{aligned}
\partial_a \gamma_7^a{}_{0i} &= \epsilon_i^{jk} \Upsilon_{ak} \gamma_7^a{}_{0j} + \epsilon_i^{jk} A_{a0k} \gamma_8^a{}_j + \frac{1}{2} \epsilon_i^{jk} F_{abk} \Gamma_3^{ab}{}_{0j} + \frac{1}{2} \epsilon_i^{jk} F_{ab0k} \Gamma_4^{ab}{}_j \\
\partial_a \gamma_8^{ai} &= \epsilon^{ijk} \Upsilon_{ak} \gamma_8^a{}_j - \epsilon^{ijk} A_{a0k} \gamma_7^a{}_{0j} - \frac{1}{2} \epsilon^{ijk} F_{ab0k} \Gamma_3^{ab}{}_{0j} + \frac{1}{2} \epsilon^{ijk} F_{abk} \Gamma_4^{ab}{}_j,
\end{aligned}$$

hence, there are 42 independent first class constraints. Therefore, by performing the counting of physical degrees of freedom we conclude that the theory is devoid of degrees of freedom, the theory

is a topological one as expected.

The algebra between the constraints is given by

$$\begin{aligned}
\{\Gamma_1^{a0i}(x), \Gamma_3^{de0l}(y)\} &= \frac{\Xi}{2} \eta^{ade} \eta^{il} \delta^3(x-y), \\
\{\Gamma_2^{ai}(x), \Gamma_4^{del}(y)\} &= -\frac{\Xi}{2} \eta^{ade} \eta^{il} \delta^3(x-y), \\
\{\gamma_5^i(x), \gamma_5^l(y)\} &= \frac{1}{2} \epsilon^{il} \gamma_5^j \delta^3(x-y) \approx 0, \\
\{\gamma_5^i(x), \gamma_6^{0l}(y)\} &= \frac{1}{2} \epsilon^{il} \gamma_5^j \delta^3(x-y) \approx 0, \\
\{\gamma_{5\ i}(x), \gamma_{7\ 0l}^d(y)\} &= \frac{1}{2} \epsilon_{il}^k \gamma_{7\ 0k}^d - \frac{1}{4\Xi} \eta_{abc} \Gamma_{4\ l}^{bc} \Gamma_{3\ 0i}^{da} + \frac{1}{4\Xi} \eta_{abc} \Gamma_{3\ l}^{bc0} \Gamma_{4\ i}^{da} \\
&\approx 0, \\
\{\gamma_{5\ i}(x), \gamma_8^{dl}(y)\} &= \frac{1}{2} \epsilon_i^{lk} \gamma_{8\ k}^d - \frac{1}{4\Xi} \eta_{abc} \Gamma_{3\ 0i}^{bc0l} \Gamma_{3\ 0k}^{da} + \frac{1}{4\Xi} \delta_i^l \eta_{abc} \Gamma_{3\ 0k}^{bc0} \Gamma_{3\ 0k}^{da} \\
&\quad + \frac{1}{4\Xi} \eta_{abc} \delta_i^l \Gamma_{4\ k}^{bck} \Gamma_{4\ k}^{da} - \frac{1}{4\Xi} \eta_{abc} \Gamma_{4\ i}^{bcl} \Gamma_{4\ i}^{da} \\
&\approx 0, \\
\{\gamma_6^{0i}(x), \gamma_6^{0l}(y)\} &= -\frac{1}{2} \epsilon^{ilj} \gamma_{5\ j} \delta^3(x-y) \approx 0, \\
\{\gamma_6^{0i}(x), \gamma_{7\ 0l}^d(y)\} &= \frac{1}{2} \epsilon_i^{lk} \gamma_{8\ k}^d - \frac{1}{4\Xi} \delta_i^l \eta_{abc} \Gamma_{3\ k}^{bc0} \Gamma_{3\ 0k}^{da} + \frac{1}{4\Xi} \eta_{abc} \Gamma_{3\ l}^{bc0} \Gamma_{3\ 0i}^{da} \\
&\quad + \frac{1}{4\Xi} \eta_{abc} \delta_i^l \Gamma_{4\ k}^{bck} \Gamma_{4\ k}^{da} - \frac{1}{4\Xi} \eta_{abc} \Gamma_{4\ l}^{bcl} \Gamma_{4\ i}^{da} \\
&\approx 0, \\
\{\gamma_6^{0i}(x), \gamma_8^{dl}(y)\} &= -\frac{1}{2} \epsilon_{ilj} \gamma_{7\ 0j}^d + \frac{1}{4\Xi} \eta_{abc} g^{il} \Gamma_{3\ 0k}^{bc0} \Gamma_{4\ k}^{da} - \frac{1}{4\Xi} \eta_{abc} \Gamma_{3\ 0l}^{bc0} \Gamma_{4\ i}^{da} \\
&\quad - \frac{1}{4\Xi} \eta_{abc} \Gamma_{4\ l}^{bcl} \Gamma_{3\ 0i}^{da} + \frac{1}{4\Xi} g^{il} \eta_{abc} \Gamma_{4\ k}^{bck} \Gamma_{3\ k}^{da0} \\
&\approx 0, \\
\{\gamma_{7\ 0i}^a(x), \Gamma_1^{d0l}(y)\} &= -\frac{1}{2} \epsilon_i^{lj} p_j^{ad} \delta^3(x-y) \\
&= -\frac{1}{2} \epsilon_i^{lj} \Gamma_{4\ j}^{ad} \delta^3(x-y) \approx 0, \\
\{\gamma_{7\ 0i}^a(x), \Gamma_2^{dl}(y)\} &= -\frac{1}{2} \epsilon_i^{lj} p_{0j}^{ad} \delta^3(x-y) \\
&= -\frac{1}{2} \epsilon_i^{lj} \Gamma_{3\ 0j}^{ad} \delta^3(x-y) \approx 0, \\
\{\gamma_8^{ai}(x), \Gamma_1^{d0l}(y)\} &= \frac{1}{2} \epsilon^{ilj} p_{0j}^{ad} \\
&= \frac{1}{2} \epsilon^{ilj} \Gamma_{3\ 0j}^{ad} \approx 0, \\
\{\gamma_8^{ai}(x), \Gamma_2^{dl}(y)\} &= \frac{1}{2} \epsilon^{ilj} p_j^{ad} \\
&= \frac{1}{2} \epsilon^{ilj} \Gamma_{4\ j}^{ad} \approx 0,
\end{aligned} \tag{69}$$

where we observe that the algebra is closed and it obeys the rules of the canonical formalism of the algebra between the constraints [16, 17, 21], namely, the result of the Poisson brackets between first class constraints with first class must be linear in first class constraints and square in second class; the result of the Poisson brackets between first class constraints with second class constraints must be linear in first class constraints and linear in second class constraints. We can observe that

our results are in full agreement with these rules. Furthermore, the constraints $\gamma_{5\ i}$ and γ_6^{0i} are identified as generators of rotations and boost respectively whereas γ_7^{0i} and γ_8^{ai} are generators of translations, this can be seen from the algebra between these constraints.

On the other hand, first class constraints are generators of gauge transformations. Hence, by defining the gauge generator in terms of first class constraints

$$G = \int [\theta_{1,i}\gamma_1^i + \theta_{2,i}\gamma_2^i + \theta_{3,ai}\gamma_3^{ai} + \theta_{4,ai}\gamma_4^{ai} + \theta_5^i\gamma_{5,i} + \theta_{6,0i}\gamma_6^{0i} + \theta_{7,a}^{0i}\gamma_{7,0i}^a + \theta_{8,ai}\gamma_8^{ai}] d^3x,$$

we find the following gauge transformations

$$\delta A_{d0l} = \{A_{d0l}, G\} \quad (70)$$

$$= \frac{1}{2} \left[-\partial_d \theta_{6,0l} + \theta_5^i \epsilon_{lik} A_{d0}^k - \epsilon_{lk}^i \theta_{6,0i} \Upsilon_d^k + \frac{1}{2\Xi} \eta_{dbc} \epsilon_{lik} \theta_5^i p^{bc0k} + \frac{1}{2\Xi} \eta_{dbc} \epsilon_{lk}^i p^{bck} \right]$$

$$\delta \Upsilon_{dl} = \{\Upsilon_{dl}, G\} \quad (71)$$

$$= \frac{1}{2} \left[-\partial_d \theta_{5,l} + \epsilon_{li}^k \theta_5^i \Upsilon_{dk} - \epsilon_l^{ij} \theta_{6,0i} A_{d0l} - \frac{1}{2\Xi} \eta_{dbc} \epsilon_{lik} \theta_5^i p^{bck} + \frac{1}{2\Xi} \eta_{dbc} \epsilon_{lk}^i \theta_{6,0i} p^{bck} \right]$$

$$\delta T_l = \{T_l, G\} = \theta_{1,l} \quad (72)$$

$$\delta \Lambda_i = \{\Lambda_i, G\} = \theta_{2,l} \quad (73)$$

$$\delta \varsigma_{dl} = \{\varsigma_{dl}, G\} = \theta_{3,dl} \quad (74)$$

$$\delta \chi_{dl} = \{\chi_{dl}, G\} = \theta_{4,dl} \quad (75)$$

$$\delta B_{de0l} = \{B_{de0l}, G\}$$

$$= \frac{1}{4} \left[\partial_e \theta_{7,d0l} - \partial_d \theta_{7,e0l} - \theta_l^{ij} \theta_{7,d0j} \Upsilon_{ek} + \epsilon_l^{ij} \theta_{7,e0i} \Upsilon_{dj} - \epsilon_l^{ik} \theta_{8,di} A_{e0k} + \epsilon_l^{ik} \theta_{8,ei} A_{d0k} \right. \\ \left. - \frac{1}{\Xi} \eta_{dea} \epsilon_{lij} \theta_5^i p^{a0j} + \frac{1}{\Xi} \eta_{dea} \epsilon_l^i \theta_{6,0i} \pi^{aj} \right] \quad (76)$$

$$\delta B_{del} = \{B_{del}, G\}$$

$$= \frac{1}{4} \left[\partial_e \theta_{8,dl} - \partial_d \theta_{8,el} - \epsilon_l^{ik} \theta_{8,di} \Upsilon_{ek} + \epsilon_l^{ik} \theta_{8,ei} \Upsilon_{dk} - \epsilon_{li}^k \theta_{7,d}^{0i} A_{e0k} + \epsilon_{li}^k \theta_{7,e}^{0i} A_{d0k} \right. \\ \left. + \frac{1}{\Xi} \eta_{dea} \epsilon_{lij} \theta_5^i \pi^{aj} + \frac{1}{\Xi} \eta_{dea} \epsilon_l^i \theta_{6,0i} p^{a0j} \right] \quad (77)$$

$$\delta p^{d0l} = \{p^{d0l}, G\}$$

$$= \frac{1}{2} \left[\Xi \eta^{dab} \partial_b \theta_{7,a}^{0l} + \epsilon^l_{ij} \theta_5^i p^{d0j} - \epsilon^{lij} \theta_{6,0i} \pi^d_k - \epsilon^l_j \theta_{7,a}^{0i} p^{ad}_j + \epsilon^{lij} \theta_{8,ai} p^{ad}_{0j} \right. \\ \left. - \Xi \eta^{dac} \epsilon^l_i \theta_{7,a}^{0i} \Upsilon_{ck} + \Xi \eta^{dac} \epsilon^{lik} \theta_{8,ai} A_{c0k} \right] \quad (78)$$

$$\delta \pi^{dl} = \{\pi^{dl}, G\}$$

$$= \frac{1}{2} \left[\Xi \eta^{dab} \partial_b \theta_{8,a}^l + \epsilon^l_i \theta_5^i \pi^d_j + \epsilon^{li}_j \theta_{6,0i} p^{d0j} + \epsilon^l_i \theta_{7,a}^{0i} p^{da}_{0j} + \epsilon^{lij} \theta_{8,ai} p^{da}_j \right. \\ \left. - \Xi \eta^{dab} \epsilon^l_i \theta_{7,a}^{0i} A_{b0j} - \Xi \eta^{dac} \epsilon^{lik} \theta_{8,ai} \Upsilon_{ck} \right] \quad (79)$$

$$\delta \hat{T}^l = \{\hat{T}^l, G\} = 0, \quad (80)$$

$$\delta \hat{\Lambda}^l = \{\hat{\Lambda}^l, G\} = 0, \quad (81)$$

$$\delta \hat{\varsigma}^{dl} = \{\hat{\varsigma}^{dl}, G\} = 0, \quad (82)$$

$$\delta \hat{\chi}^{dl} = \{\hat{\chi}^{dl}, G\} = 0, \quad (83)$$

$$\delta p^{de0l} = \{p^{de0l}, G\} = 0, \quad (84)$$

$$\delta p^{del} = \{p^{del}, G\} = 0. \quad (85)$$

Now, with the second class constraints we can calculate the Dirac brackets. In order to perform this aim we calculate the matrix $C^{\alpha\beta} = \{\Gamma^\alpha, \Gamma^\beta\}$ whose entries are given by the Poisson brackets of the second class constraints, namely

$$C_{\alpha\beta} = \begin{pmatrix} \{\Gamma_1^{a0i}, \Gamma_1^{d0l}\} & \{\Gamma_1^{a0i}, \Gamma_2^{dl}\} & \{\Gamma_1^{a0i}, \Gamma_3^{de0l}\} & \{\Gamma_1^{a0i}, \Gamma_4^{del}\} \\ \{\Gamma_2^{ai}, \Gamma_1^{d0l}\} & \{\Gamma_2^{ai}, \Gamma_2^{dl}\} & \{\Gamma_2^{ai}, \Gamma_3^{de0l}\} & \{\Gamma_2^{ai}, \Gamma_4^{del}\} \\ \{\Gamma_3^{ab0i}, \Gamma_1^{d0l}\} & \{\Gamma_3^{ab0i}, \Gamma_2^{dl}\} & \{\Gamma_3^{ab0i}, \Gamma_3^{de0l}\} & \{\Gamma_3^{ab0i}, \Gamma_4^{del}\} \\ \{\Gamma_4^{abi}, \Gamma_1^{d0l}\} & \{\Gamma_4^{abi}, \Gamma_2^{dl}\} & \{\Gamma_4^{abi}, \Gamma_3^{de0l}\} & \{\Gamma_4^{abi}, \Gamma_4^{del}\} \end{pmatrix} \\ = \begin{pmatrix} 0 & 0 & \frac{\Xi}{2}\eta^{ade}\eta^{il} & 0 \\ 0 & 0 & 0 & -\frac{\Xi}{2}\eta^{ade}\eta^{il} \\ -\frac{\Xi}{2}\eta^{abd}\eta^{il} & 0 & 0 & 0 \\ 0 & \frac{\Xi}{2}\eta^{abd}\eta^{il} & 0 & 0 \end{pmatrix} \delta^3(x-y), \quad (86)$$

and the inverse of $C_{\alpha\beta}$ is given by

$$C_{\alpha\beta}^{-1} = \begin{pmatrix} 0 & 0 & -\frac{2}{\Xi}\eta_{ade}\eta_{il} & 0 \\ 0 & 0 & 0 & \frac{2}{\Xi}\eta_{ade}\eta_{il} \\ \frac{1}{\Xi}\eta_{abd}\eta_{il} & 0 & 0 & 0 \\ 0 & -\frac{1}{\Xi}\eta_{abd}\eta_{il} & 0 & 0 \end{pmatrix} \delta^3(x-y). \quad (87)$$

Hence, the Dirac brackets between two functionals, namely $F(q, p)$ and $G(q, p)$, is defined by

$$\{F, G\}_D \equiv \{F, G\} - \int dudv \{F, \Gamma^\alpha(u)\} C_{\alpha\beta}^{-1}(u, v) \{\Gamma^\beta(v), G\}, \quad (88)$$

where Γ^α represent the second class constraints. Therefore, the Dirac brackets between the fields are given by

$$\{A_{a0i}(x), p^{d0j}(y)\}_D = \{A_{a0i}(x), p^{d0j}(y)\} = \frac{1}{2}\delta_a^d \delta_i^j \delta^3(x-y), \quad (89)$$

$$\{\Upsilon_{ai}(x), \pi^{dj}(y)\}_D = \{\Upsilon_{ai}(x), \pi^{dj}(y)\} = \frac{1}{2}\delta_a^d \delta_i^j \delta^3(x-y), \quad (90)$$

$$\{T_i(x), \hat{T}^j(y)\}_D = \{T_i(x), \hat{T}^j(y)\} = \frac{1}{2}\delta_i^j \delta^3(x-y), \quad (91)$$

$$\{\Lambda_i(x), \hat{\Lambda}^j(y)\}_D = \{\Lambda_{a0i}(x), \hat{\Lambda}^{d0j}(y)\} = \frac{1}{2}\delta_i^j \delta^3(x-y), \quad (92)$$

$$\{\varsigma_{ai}(x), \varsigma^{dj}(y)\}_D = \{\varsigma_{ai}(x), \varsigma^{dj}(y)\} = \delta_a^d \delta_i^j \delta^3(x-y), \quad (93)$$

$$\{\chi_{ai}(x), \hat{\chi}^{dj}(y)\}_D = \{\chi_{ai}(x), \hat{\chi}^{dj}(y)\} = \delta_a^d \delta_i^j \delta^3(x-y), \quad (94)$$

$$\{B_{ab0i}(x), p^{de0j}(y)\}_D = 0, \quad (95)$$

$$\{B_{abi}(x), p^{dej}(y)\}_D = 0, \quad (96)$$

$$\{A_{a0i}(x), B_{de0j}(y)\}_D = -\frac{1}{2\Xi}\eta_{ade}\eta_{ij}\delta^3(x-y), \quad (97)$$

$$\{\Upsilon_{ai}(x), B_{dej}(y)\}_D = \frac{1}{2\Xi}\eta_{ade}\eta_{ij}\delta^3(x-y), \quad (98)$$

In addition, we can also calculate the Dirac brackets by gauge fixing, in this case we will use the

temporal gauge, namely, we take

$$T_i \approx 0, \quad (99)$$

$$\Lambda_i \approx 0, \quad (100)$$

$$\varsigma_{ai} \approx 0, \quad (101)$$

$$\chi_{ai} \approx 0. \quad (102)$$

These conditions are considered as a new set of second class constraints, hence, now there are the following 84 second class constraints

$$\begin{aligned} \Gamma_1^{a0i} &\equiv p^{a0i} - \Xi \eta^{abc} B_{bc}{}^{0i} \approx 0, \\ \Gamma_2^{ai} &\equiv \pi^{ai} - \Xi \eta^{abc} B_{bc}{}^i \approx 0, \\ \Gamma_3^{ab0i} &\equiv p^{ab0i} \approx 0, \\ \Gamma_4^{abi} &\equiv p^{abi} \approx 0, \\ \Gamma_{5i} &\equiv T_i \approx 0, \\ \Gamma_{6i} &\equiv \Lambda_i \approx 0, \\ \Gamma_{7ai} &\equiv \varsigma_{ai} \approx 0, \\ \Gamma_{8ai} &\equiv \chi_{ai} \approx 0, \\ \Gamma_{9i} &\equiv \hat{T}_i \approx 0, \\ \Gamma_{10i} &\equiv \hat{\Lambda}_i \approx 0, \\ \Gamma_{11ai} &\equiv \hat{\varsigma}_{ai} \approx 0, \\ \Gamma_{12ai} &\equiv \hat{\chi}_{ai} \approx 0. \end{aligned} \quad (103)$$

In this manner, the matrix $C_{\alpha\beta} = \{\Gamma^\alpha, \Gamma^\beta\}$ whose entries are given by the Poisson brackets between the second class constraints (103) is given by

$$C_{\alpha\beta} = \begin{pmatrix} 0 & 0 & \frac{\Xi}{2} \eta^{ade} \eta^{il} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\Xi}{2} \eta^{ade} \eta^{il} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\Xi}{2} \eta^{abd} \eta^{il} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\Xi}{2} \eta^{abd} \eta^{il} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \delta_i^l & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \delta_i^l & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_a^d \delta_i^l & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_a^d \delta_i^l & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \delta_l^i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \delta_l^i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\delta_d^a \delta_l^i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta_d^a \delta_l^i & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \delta^3(x-y), \quad (104)$$

and its inverse

$$C_{\alpha\beta}^{-1} = \begin{pmatrix} 0 & 0 & -\frac{2}{\Xi}\eta_{ade}\eta_{il} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{\Xi}\eta_{ade}\eta_{il} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\Xi}\eta_{abd}\eta_{il} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\Xi}\eta_{abd}\eta_{il} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\delta_l^i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\delta_l^i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta_a^d\delta_l^i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta_a^d\delta_l^i & 0 \\ 0 & 0 & 0 & 0 & 2\delta_i^l & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\delta_i^l & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \delta_a^d\delta_i^l & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_a^d\delta_i^l & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \delta^3(x-y). \quad (105)$$

In this manner, by using (105) we construct the Dirac brackets given by

$$\begin{aligned} \{A_{a0i}(x), p^{d0j}(y)\}_D &= \{A_{a0i}(x), p^{d0j}(y)\} = \frac{1}{2}\delta_a^d\delta_i^j\delta^3(x-y), \\ \{\Upsilon_{ai}(x), \pi^{dj}(y)\}_D &= \{\Upsilon_{ai}(x), \pi^{dj}(y)\} = \frac{1}{2}\delta_a^d\delta_i^j\delta^3(x-y), \\ \{T_i(x), \hat{T}^j(y)\}_D &= 0, \\ \{\Lambda_i(x), \hat{\Lambda}^j(y)\}_D &= 0, \\ \{\varsigma_{ai}(x), \hat{\varsigma}^{dj}(y)\}_D &= 0, \\ \{\chi_{ai}(x), \hat{\chi}^{dj}(y)\}_D &= 0, \\ \{B_{ab0i}(x), p^{de0j}(y)\}_D &= 0, \\ \{B_{abi}(x), p^{dej}(y)\}_D &= 0, \\ \{A_{a0i}(x), B_{de0j}(y)\}_D &= -\frac{1}{2\Xi}\eta_{ade}\eta_{ij}\delta^3(x-y), \\ \{\Upsilon_{ai}(x), B_{dej}(y)\}_D &= \frac{1}{2\Xi}\eta_{ade}\eta_{ij}\delta^3(x-y), \end{aligned} \quad (106)$$

where we can observe that the Dirac brackets and the generalized FJ brackets coincide to each other.

IV. CONCLUSIONS AND PROSPECTS

In this paper, the symplectic analysis of a four-dimensional BF theory has been performed. We reported the complete set of FJ constraints and we observe that there are present less constraints than in Dirac's method. Furthermore, we have carried out the counting of physical degrees of freedom concluding that the theory is a topological one as expected. In addition, we have used a temporal gauge in order to obtain a symplectic tensor, then the quantization brackets of FJ were obtained. On the other hand, a pure canonical analysis has been performed. Under a laborious work, we have reported the complete set of first class and second class constraints, the algebra between the constraints is in full agreement with the canonical rules; then using a temporal gauge the Dirac brackets were computed. The FJ and Dirac's brackets coincide to each other, thus we can conclude that the FJ is more economical than Dirac framework. Of course, if in the

Dirac approach are introduced the Dirac brackets and the second class constraints are considered as strong equations, then the FJ and Dirac's constraints coincide to each other. Finally we would like to comment that in this work we provide the necessary tools for studying in alternative way the BF formulations of gravity such as that reported in [22]. In fact, in that work the canonical formulation of BF gravity has been performed, and will be interesting to study that theory by using the ideas of the symplectic formalism of FJ. All these ideas are in progress and will be the subject of forthcoming works.

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